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Elastic constants for granular materials modeled as first-order strain-gradient continua

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Abstract

Formulation of a stress–strain relationship is presented for a granular medium, which is modeled as a first-order strain-gradient continuum. The elastic constants used in the stress–strain relationship are derived as an explicit function of inter-particle stiffness, particle size, and packing density. It can be demonstrated that couple-stress continuum is a subclass of strain-gradient continua. The derived stress–strain relationship is simplified to obtain the expressions of elastic constants for a couple-stress continuum. The derived stress–strain relationship is compared with that of existing theories on strain- gradient models. The effects of inter-particle stiffness and particle size on material constants are discussed.

Keywords: Granular material; Stress–strain relationship; Internal length; Strain-gradient model; Couple stress model; Grain rotation

1. Introduction

A high-gradient model utilizes higher-order derivatives of displacement as additional strain measures in a constitutive equation. Therefore, the model is useful in simulating materials with high intensity of non-uniform strains. For this reason, high-gradient models have been recently received attention in the analysis of several phenomena where strains are highly non-uniform such as: localized deformation including shear band or fracture (Coleman and Hodgdon, 1985; Triantafyllidis and Aifantis, 1986; Bazant and Pijaudier-Cabot, 1987; Bardenhagen and Trianfyllidis, 1994; Zervos et al., 2001) propagation of high-frequency waves in a media (Chang and Gao, 1997; Sluys, 1992; Suiker et al., 1999) and mesh dependency in finite element analysis for materials in a near-failure condition (De Borst and Muhlhaus, 1992; Pamin, 1944; Peerlings et al., 1996).

Two types of high-gradient models can be found in the literature, namely, (1) models employing only higher-order strain, and (2) models employing both higher-order stress and higher-order strain. For models

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employing only higher-order strain, Cauchy stress is a function of strain as well as higher-order strain. However, higher-order stress is not considered. The concept of this model can be traced to non-local theories in which the magnitude of stress can be influenced by the strains at the vicinity of a material point. This type of model is popularly used because the equilibrium equation for this type of model is identical to that used for the classic model, therefore the conventional solution methods can be directly utilized without much alternation. For this type of model, several expressions for elastic stress–strain relationships have been proposed, e.g., Beran and McCoy (1970), Bazant and Pijaudier-Cabot (1987), Altan and Aifantis (1997), Chang and Gao (1997), Muhlhaus and Oka (1996). This approach, however, has an unfavorable aspect. Because the higher-order stresses are neglected, models employing only higher-order strains become ambiguous in the definition of work done due to higher-order strain. Therefore, a numerical implementation of this type of model may encounter problems associated with non-positive definiteness.

On the other hand, for models employing both higher-order stress and higher-order strain, the higher-order stress is the energy counter part of the strain-gradients. Thus a positive potential energy is ensured. This type of model can be found in the earlier work by Mindlin (1965), Toupin (1962), etc. The equilibrium equation for the system involves not only Cauchy stress but also the high-order stresses. As a result, it greatly increases the complexity in solution methods. For example, the shape functions used in a finite element method are required to satisfy the continuities of not only displacement but also displacement gradients. Moreover, it will also increase the number of material constants. For example, in addition to the usual two Lame constants, Mindlin and Eshel's model (1968) consists of five other material constants for isotropic materials. Unfortunately, these constants are difficult to be determined, through either experimental means or otherwise. This difficulty is the primary factor that prohibits investigators from adopting the use of higher-order stresses.

In this paper, we attempt to derive expressions of material constants for granular material, which is modeled as a strain-gradient continuum employing both higher-order stress and higher-order strain. Here, we will focus only on elastic constants. It is worthwhile to examine the expressions of material constants for the simple linear elastic conditions because non-uniform strains are often observed in granular material even in the elastic range, and an elastic stress–strain relationship is the basic element for a comprehensive elastic–plastic-damage model.

For this purpose, we will extend the granular mechanics approach (i.e. Chang and Liao, 1990; Chang and Gao, 1995) to include the effects of higher-order stress. In Section 2, we will derive the stress–strain relationship. The essential ingredients of the model such as continuous displacement field, inter-particle stiffness, and fabric tensor are discussed sequentially. In Section 3, closed-form expressions for the elastic constants are derived for randomly packed granulates. The derived stress–strain expressions are then compared to those of existing theories. In Section 4, the derived stress–strain relationships are simplified to a form that resembles a couple-stress model. The effects of particle size, inter-particle stiffness, and size of representative volume on elastic constants are discussed. Summary and conclusions are given in Section 5.

2. Stress–strain law for the first-order strain-gradient continua

In nature, all materials are not continuous if the viewing scale is sufficiently small. However, for practical purposes, it is convenient to model the mechanical behavior of a material by treating it as a continuum. In the mechanics of classic continuum, the stress–strain relationship is defined for a "volume element" which is conceptually infinitesimal in size. However, the size of a volume element should be relatively larger than the size of material microstructure so that the volume element can be treated as a continuum. For a randomly packed granular material, the volume element should contain sufficient number of particles in order to be representative of the material and to resemble a continuum. Therefore, the size of a representative volume element envisioned for a granular material is much larger than that for a metallic material.

Treating the representative volume of granular material as a continuum, stress can be linked to inter-particle forces and strain can be linked to particle movements. Works along this line can be found by Rothenburg and Selvadurai (1981), Wallton (1987), Jenkins (1988), Chang (1988), Chang and Gao (1996), Kruyt and Rothenburg (2001, 2002), Luding et al. (2001), Kruyt (2003), etc. In order to model high intensities of non-uniform deformation of granular material, it is desirable to include strain gradients as an additional strain measure, and model the material as a strain-gradient continuum. The material constants derived for models employing only higher-order strain can be found in the work by Chang and Gao (1995), Chang (1998), Muhlhaus and Oka (1996), Suiker et al. (2001), etc. However, very little work has been focused on material constants for models that employ both higher-order stress and higher-order strain.

2.1. Continuum field for a discrete particle system

Granular material is envisioned as a collection of particles. Under deformation, particles in the material undergo translation and rotation. In order to make a link between the discrete particle system and its equivalent continuum system, we construct a continuum displacement field $u_i(\boldsymbol{x})$ in such a way that the displacement at the centroid of the nth particle, u_i^n, coincides with the displacement field, i.e,

$$u_i(\mathbf{x}^n) = u_i^n \tag{1}$$

where $\mathbf{x}^n$ is the location of the nth particle.

In classic continuum mechanics, a linear displacement field is employed to describe the deformation of a representative volume element. For granular material, the size of a representative volume element is relatively large. Thus we approximate the displacement field by a polynomial expansion containing second derivatives, i.e.,

$$u_i(\mathbf{x}) = u_i + u_{i,j}x_j + \tfrac{1}{2}u_{i,jk}x_jx_k \tag{2}$$

where u_i, $u_{i,j}$, and $u_{i,jk}$ are constants for the representative volume.

The second-rank tensor $u_{i,j}$ has nine components. The third-rank tensor $u_{i,jk}$ has 27 components. However, by definition the last two indices of $u_{i,jk}$ (i.e., j and k) are reciprocal. Thus the 27 components can be reduced to only 18 independent components.

For simplicity, we assume the rotation of particles is equal to the rigid-body rotation of the representative volume and there is no moment transmitting through the contact between particles. The material can be regarded as the Class II non-polar type of continuum (Chang and Gao, 1995).

2.2. Inter-particle contact law

We now consider the elastic behavior of two particles in contact. A general expression for the constitutive relations between the inter-particle force f_q^c and the relative displacement of two particles δ_k^c (i.e., inter-particle compression) can be given by

$$f_q^c = K_{qk}^c\delta_k^c \tag{3}$$

where K_{qk}^c is the inter-particle contact stiffness tensor. For two particles in contact, a local coordinate system can be constructed for each contact with three orthogonal base unit vectors: n_k is normal to the contact plane; s_k and t_k are tangential to the contact plane as shown in Fig. 1.

Let k_n be the compressive contact stiffness in normal direction and k_s be the shear contact stiffness. Assuming the shear contact stiffness is same in **s** and **t** directions and that there is no coupling effect between normal and shear directions, the contact stiffness tensor K_{qk}^c can then be expressed in terms of the unit vectors **n**, **s** and **t**, as

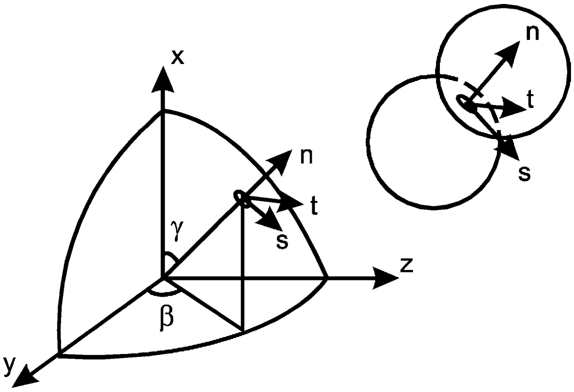

Fig. 1. Local coordinate at inter-particle contact.

$$K^c_{qk} = k_n n^c_q n^c_k + k_s \left(s^c_q s^c_k + t^c_q t^c_k \right) \tag{4}$$

For each particle contact, the corresponding auxiliary local coordinate system is related to the global coordinate system according to (see Fig. 1)

$$\begin{aligned} \vec{n} &= (\cos\gamma, \sin\gamma\cos\beta, \sin\gamma\sin\beta) \\ \vec{s} &= (-\sin\gamma, \cos\gamma\cos\beta, \cos\gamma\sin\beta) \\ \vec{t} &= (0, -\sin\beta, \cos\beta) \end{aligned} \tag{5}$$

The vector $\vec{s}$ is on the plane consisting of $\vec{x}$ and $\vec{n}$. The vector $\vec{t}$ is perpendicular to this plane and can be obtained by the cross product of $\vec{n} \times \vec{s}$. The rolling resistances between two particles are neglected in this paper.

2.3. Fabric tensors

For convenience in the further derivation, we define fabric tensors l^c_i and J^c_{ij} as follows:

$$l^c_i = x^b_i - x^a_i; \quad J^c_{ij} = x^b_i x^b_j - x^a_i x^a_j \tag{6}$$

The first order fabric tensor (or branch vector), l^c_i, represents the vector from the centroid of particle 'a' to that of particle 'b' as shown in Fig. 2. Let the position vector of the contact point be x^c_i, thus

$$x^a_i = x^c_i - r^{ac}_i; \quad x^b_i = x^c_i - r^{bc}_i \tag{7}$$

By substituting Eqs. (7) into (6), the second order fabric tensor, J^c_{ij}, can be expressed as functions of the location of contact, x^c_i, and the branch vector l^c_i as follows (note that $l^c_i = r^{ac}_i - r^{bc}_i$):

$$J^c_{ij} = \left(x^c_i - r^{bc}_i\right)\left(x^c_j - r^{bc}_j\right) - \left(x^c_i - r^{ac}_i\right)\left(x^c_j - r^{ac}_j\right) = x^c_i l^c_j + x^c_j l^c_i + \left(r^{bc}_i r^{bc}_j - r^{ac}_i r^{ac}_j\right) \tag{8}$$

The second order tensor is a symmetric tensor, which includes descriptions of branch vector, location of the contact, and radii of the two associated particles. In the special case of a packing with equal-size spheres, the second order tensor $J^c_{ij} = x^c_i l^c_j + x^c_j l^c_i$.

Between the two individual convex-shaped particles as shown in Fig. 2, the inter-particle compression δ^c_i at the contact point c can be obtained as follows:

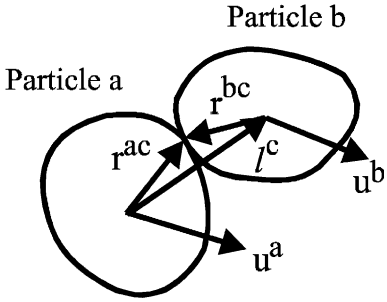

Fig. 2. Schematic plot of two particles in contact.

$$\delta_i^c = \delta_i^{ab} = u_i(\mathbf{x}^b) - u_i(\mathbf{x}^a) + e_{ijk}\left(\omega_j^b r_k^{bc} - \omega_j^a r_k^{ac}\right) \tag{9}$$

where ω_j^b and ω_j^a are respectively the rotations of particle 'b' and particle 'a'. As previously described, the particle rotation is assumed to be equal to the rigid-body rotation (i.e., $\omega_j^b = \omega_j^a = \omega_j$) of the representative volume element, which can be expressed as the skew-symmetric part of strain

$$e_{ijk}\omega_j = u_{[i,k]} = (u_{i,k} - u_{k,i})/2 \tag{10}$$

Thus Eq. (9) becomes

$$\delta_i^c = \delta_i^{ab} = u_i(\mathbf{x}^b) - u_i(\mathbf{x}^a) - e_{ijk}\omega_j l_k^c \tag{11}$$

Using the polynomial expression in Eq. (2), we can describe the inter-particle compression using the continuum field of displacement.

$$\delta_i^c = u_{i,j} l_j^c + \tfrac{1}{2} u_{i,jk} J_{jk}^c + \cdots - e_{ijk}\omega_j l_k^c \tag{12}$$

2.4. Constitutive equation

Let the representative volume V be subjected to forces on its boundary surface S. Neglecting the body force of particles, the work done per unit volume of the discrete system can be expressed as a summation of the work done over all inter-particle contacts in the unit volume. Then, by substituting the inter-particle compression δ_i^c with Eq. (12), it yields

$$W = \frac{1}{V}\sum_V f_i^c \delta_i^c = \frac{1}{V}\sum_V \left(\left(u_{i,k} - e_{ijk}\omega_j\right) f_i^c l_k^c + \frac{1}{2} u_{i,jk} f_i^c J_{jk}^c\right) \tag{13}$$

It can be easily observed that the term $u_{i,k} - e_{ijk}\omega_j$ is equal to the symmetric strain $u_{(i,k)} = (u_{i,k} + u_{k,i})/2$. The work done per unit volume in Eq. (13) can be rearranged into

$$W = \sigma_{iq} u_{(q,i)} + \sigma_{ijq} u_{q,ij} \tag{14}$$

where the stresses are defined in terms of fabric tensors as follows

$$\sigma_{iq} = \frac{1}{V}\sum_V f_q^c l_i^c; \quad \sigma_{ijq} = \frac{1}{2V}\sum_V f_q^c J_{ij}^c \tag{15}$$

Eq. (14) shows that the work done on the system is contributed from two terms: the symmetric strain $u_{(q,i)}$ and the first-gradient of strain $u_{q,ij}$. The expression of Eq. (14) is similar to the form of first-order strain-gradient theories proposed by Fleck and Huchinson (1993), Mindlin and Eshel (1968), Mindlin and

Tiersten (1962), Toupin (1962), etc. Using the inter-particle contact law (Eq. (3)), and the continuum polynomial expression of δ_i^c (Eq. (12)), Eq. (15) can be further expressed as

$$\begin{aligned}\sigma_{iq} &= C_{iqkl}u_{(k,l)} + B_{iqklm}u_{k,lm}\\ \sigma_{ijq} &= B_{lqkij}u_{(k,l)} + D_{ijqklm}u_{k,lm}\end{aligned} \tag{16}$$

where the constitutive coefficient tensors are expressed in terms of fabric tensors and inter-particle stiffness:

$$C_{iqkl} = \frac{1}{V}\sum_{c=1}^{N} l_i^c l_l^c K_{qk}^c; \quad B_{iqklm} = \frac{1}{2V}\sum_{c=1}^{N} l_i^c J_{lm}^c K_{qk}^c; \quad D_{ijqklm} = \frac{1}{4V}\sum_{c=1}^{N} J_{ij}^c J_{lm}^c K_{qk}^c \tag{17}$$

To determine the constitutive tensors B_{iqklm} and D_{ijqklm}, we first obtain the expressions for $l_i^c J_{lm}^c$ and $J_{ij}^c J_{lm}^c$ with the aid of Eq. (8)

$$l_i^c J_{lm}^c = x_l^c l_i^c l_m^c + x_m^c l_i^c l_l^c + l_i^c \left(r_l^{bc} r_m^{bc} - r_l^{ac} r_m^{ac}\right) \tag{18}$$

$$\begin{aligned} J_{ij}^c J_{lm}^c &= x_i^c x_l^c l_j^c l_m^c + x_i^c x_m^c l_j^c l_l^c + x_j^c x_l^c l_i^c l_m^c + x_j^c x_m^c l_i^c l_l^c + \left(x_j^c l_i^c + x_i^c l_j^c\right)\left(r_i^{bc} r_j^{bc} - r_i^{ac} r_j^{ac}\right)\\ &\quad + \left(x_l^c l_m^c + x_m^c l_l^c\right)\left(r_l^{bc} r_m^{bc} - r_l^{ac} r_m^{ac}\right) + \left(r_i^{bc} r_j^{bc} - r_i^{ac} r_j^{ac}\right)\left(r_l^{bc} r_m^{bc} - r_l^{ac} r_m^{ac}\right)\end{aligned} \tag{19}$$

Then, we substitute the expressions of fabric tensors $l_i^c J_{lm}^c$ and $J_{ij}^c J_{lm}^c$ into Eq. (17), which yields

$$B_{iqklm} = \frac{1}{2V}\sum_{c=1}^{N} l_i^c J_{lm}^c K_{qk}^c = \frac{1}{4V}\sum_{c=1}^{N} x_l^c l_i^c l_m^c K_{qk}^c + \frac{1}{4V}\sum_{c=1}^{N} x_m^c l_i^c l_l^c K_{qk}^c + \frac{1}{4V}\sum_{c=1}^{N} l_i^c \left(r_l^{bc} r_m^{bc} - r_l^{ac} r_m^{ac}\right) K_{qk}^c \tag{20}$$

For a representative volume of granular material, the medium can be treated as statistically homogeneous and can thus be regarded as possessing central symmetry. Because of centro-symmetry, the first two terms in Eq. (20) involving x_l^c (or x_m^c) are zero. Note that the last term is equal to zero for a packing with equal-size spheres. Nevertheless, this term is expected to be negligible since we assume the representative volume contains a sufficiently large number of particles, and the particle size is relatively small compared to the size of representative volume. Thus, $B_{iqklm} \approx 0$.

Using the same arguments, the constitutive tensor D_{ijqklm} becomes

$$D_{ijqklm} = \frac{1}{4V}\sum_{c=1}^{N} J_{ij}^c J_{lm}^c K_{qk}^c = \frac{1}{4V}\left(\sum_{c=1}^{N} x_j^c x_m^c l_i^c l_l^c + \sum_{c=1}^{N} x_i^c x_m^c l_j^c l_l^c + \sum_{c=1}^{N} x_j^c x_l^c l_i^c l_m^c + \sum_{c=1}^{N} x_i^c x_l^c l_j^c l_m^c\right) K_{qk}^c \tag{21}$$

Since $B_{iqklm} = 0$, the overall constitutive equations of Eq. (14) can be simplified into two fully decoupled equations:

$$\begin{aligned}\sigma_{iq} &= C_{iqkl}u_{(k,l)}\\ \sigma_{ijq} &= D_{ijqklm}u_{k,lm}\end{aligned} \tag{22}$$

The constitutive equations consist of one fourth-rank tensor C_{iqkl}, and one sixth-rank tensor, D_{ijqklm}. According to Wyle's theory (Suiker and Chang, 2000), most of the components of an isotropic tensor are dependent. In a general case, out of the 81 components of the fourth-rank tensor C_{iqkl}, there are only three independent constants. The sixth-rank tensor, D_{ijqklm}, contains 729 components, but only 15 independent constants. Thus for the two set of constitutive equations, we have total 18 material constants, which is still rather a formidable situation. To investigate the possibility for a further simplification, in what follows, we will analyze the properties of the higher-rank constitutive tensors D_{ijqklm}.

Since the representative volume is a packing of randomly arranged particles, it is reasonable to state that, for any pair of two particles in contact, the location of contact 'c' is not correlated to the branch orientation. According to the covariance theory of statistics (see derivation in Appendix A), the summation can be written as a product of two separate summations as follows:

$$\frac{1}{V}\sum_{c=1}^{N} x_j^c x_m^c l_i^c l_l^c K_{pk}^c = \left(\frac{1}{N}\sum_{c=1}^{N} x_j^c x_m^c\right)\left(\frac{1}{V}\sum_{c=1}^{N} l_i^c l_l^c K_{pk}^c\right) = I_{jm} C_{ipkl} \tag{23}$$

where the second summation is identical to the fourth-rank tensor C_{ipkl} and I_{jm} is the second-moment of inertia of the representative volume, defined as

$$I_{ij} = \frac{1}{N}\sum_{c=1}^{N} x_i^c x_j^c \tag{24}$$

Therefore, with the assumption that the packing is perfectly random and that the representative volume is sufficiently larger than the particle, the sixth-rank constitutive tensor can be reduced to a summation of the products of second-order moment of inertia and the fourth-rank constitutive tensor, given by

$$D_{ijpklm} = \tfrac{1}{4}\left(I_{jm} C_{ipkl} + I_{im} C_{jpkl} + I_{jl} C_{ipkm} + I_{il} C_{jpkm}\right) \tag{25}$$

This fact simplifies greatly the complexity of the higher-order constitutive tensors. The values of D_{ijqklm} can be determined directly from C_{iqkl}.

3. Elastic material constants for strain-gradient continuum

3.1. The relation to inter-particle stiffness

In this section, closed-form expressions for the constitutive tensors, C_{ijkl} and D_{ijqklm}, are derived in terms of inter-particle properties. For a representative volume that contains a sufficiently large number of randomly packed particles, the summation in Eq. (17) can be converted to an integral form. For an isotropic random packing structure, the integral is given by

$$\sum_{c=1}^{N} l_i^c l_l^c K_{qk}^c = \frac{N}{4\pi}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} l_i^c(\beta,\gamma)\, l_l^c(\beta,\gamma)\, K_{qk}^c(\beta,\gamma)\sin\gamma\, \mathrm{d}\beta\, \mathrm{d}\gamma \tag{26}$$

In this integral, the direction of branch vector is n_i^c, and the stiffness tensor is a function of n_i^c, s_i^c, t_i^c as shown in Eq. (4). The vectors n_i^c, s_i^c, t_i^c (see Fig. 1) can be replaced with continuous functions of (β,γ) as given in Eq. (5). The integral was carried out and it yielded a closed-form expression for the fourth-rank tensor C_{ijkl} in the following form: (Chang and Gao, 1995)

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \alpha\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right) \tag{27}$$

The three material constants, λ, μ, and α derived by Chang and Gao (1995) are in explicit terms of contact stiffness, k_n and k_s, branch length l, and the number of inter-particle contacts per unit volume N/V (i.e., packing density).

$$\lambda = \frac{Nl^2}{15V}(k_\mathrm{n} - k_\mathrm{s});\quad \mu = \frac{Nl^2}{15V}\left(k_\mathrm{n} + \frac{3}{2}k_\mathrm{s}\right);\quad \alpha = \frac{Nl^2}{6V}k_\mathrm{s} \tag{28}$$

In Eq. (28), note that $\alpha = \mu - \lambda$. The three material constants can thus be reduced to two independent constants; i.e., the usual Lame constants, λ and μ.

Since in the stress–strain relationship $\sigma_{ij} = C_{ijkl} u_{(k,l)}$, the strain $u_{(k,l)}$ is symmetric about subscripts k and l, the fourth-rank constitutive tensor can be simplified to

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{29}$$

Denoting the strain tensor $\varepsilon_{ij} = u_{(i,j)}$, the stress–strain equation becomes the familiar form as follows:

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \tag{30}$$

As derived in the previous section, the sixth rank tensors, D_{ijqklm}, are functions of the fourth-rank tensor C_{ijkl} and the second-moment of inertia I_{ij}. For a cubic representative volume with side length $2L$, the second-moment of inertia in Eq. (24) can be written in a form of integral, given by

$$\sum_{c=1}^{N} x_i^c x_j^c = \frac{N}{8L^3}\int_{-L}^{L}\int_{-L}^{L}\int_{-L}^{L} x_i x_j \,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 \tag{31}$$

After integration, it yields

$$I_{ij} = \frac{1}{3}L^2\delta_{ij} \tag{32}$$

By substituting the expression C_{ijkl} of Eq. (27) into Eq. (25), we obtain

$$D_{ikjqpm} = \frac{1}{4}\frac{L^2}{3}\left(\lambda\Lambda^{\lambda}_{ijkqpm} + \mu\Lambda^{\mu}_{ijkqpm} + \alpha\Lambda^{\alpha}_{ijkqpm}\right) \tag{33}$$

where

$$\Lambda^{\lambda}_{ijkqpm} = \delta_{jm}\delta_{ip}\delta_{kl} + \delta_{im}\delta_{jp}\delta_{kl} + \delta_{jl}\delta_{ip}\delta_{km} + \delta_{il}\delta_{jp}\delta_{km}$$
$$\Lambda^{\mu}_{ijkqpm} = \delta_{jm}\delta_{ik}\delta_{pl} + \delta_{jm}\delta_{il}\delta_{pk} + \delta_{im}\delta_{jk}\delta_{pl} + \delta_{im}\delta_{jl}\delta_{pk} + \delta_{jl}\delta_{ik}\delta_{pm} + \delta_{jl}\delta_{im}\delta_{pk} + \delta_{il}\delta_{jk}\delta_{pm} + \delta_{il}\delta_{jm}\delta_{pk}$$
$$\Lambda^{\alpha}_{ijkqpm} = -\delta_{jm}\delta_{ik}\delta_{pl} + \delta_{jm}\delta_{il}\delta_{pk} - \delta_{im}\delta_{jk}\delta_{pl} + \delta_{im}\delta_{jl}\delta_{pk} - \delta_{jl}\delta_{ik}\delta_{pm} + \delta_{jl}\delta_{im}\delta_{pk} - \delta_{il}\delta_{jk}\delta_{pm} + \delta_{il}\delta_{jm}\delta_{pk}$$

Eq. (33) leads to the final form of higher-order stress–stress relationship

$$\sigma_{ijp} = \frac{L^2}{6}\left(\lambda(\delta_{ip}u_{k,kj} + \delta_{jp}u_{k,ki}) + \mu(2u_{p,ij} + u_{i,pj} + u_{j,pi}) + \alpha(2u_{p,ij} - u_{i,pj} - u_{j,pi})\right) \tag{34}$$

According to Suiker and Chang (2000), 15 independent constants are required to completely define an isotropic six-rank tensor. However, by considering the two conditions for D_{ijqklm}: (a) the indices l and m are reciprocal, and (b) the indices i and j are reciprocal, the 15 constants can be reduced to three constants: λ, μ and L (note that α is a function of λ and μ).

3.2. Comparison with other formulations

It is instructive to compare the present model with that derived by Mindlin and Eshel (1968). They proposed a seven-constant general form of work done per unit volume,

$$W = \left(\mu\varepsilon_{ij}\varepsilon_{ij} + \tfrac{1}{2}\lambda\varepsilon_{kk}\varepsilon_{jj}\right) + a_1u_{k,ii}u_{j,kj} + a_2u_{j,ij}u_{k,ik} + a_3u_{k,ii}u_{k,jj} + a_4u_{k,ij}u_{k,ij} + a_5u_{k,ij}u_{i,jk} \tag{35}$$

Taking derivative of W with respect to strain and to strain-gradient, the corresponding higher-order stress–strain relationship can be derived, given by

$$\sigma_{pqr} = a_1u_{i,pi}\delta_{qr} + a_1u_{i,qi}\delta_{pr} + \tfrac{1}{2}a_2(u_{p,ii}\delta_{qr} + 2u_{i,ri}\delta_{qp} + u_{q,ii}\delta_{pr}) + 2a_3u_{r,ii}\delta_{pq} + 2a_4u_{r,pq} + a_5(u_{p,rq} + u_{q,rp}) \tag{36}$$

The five material constants a_i are difficult to be obtained from experiments. Very few formulations are available with regard to the material constants, except for some limited conditions. For example the following expression for one-dimensional stretch was used by Day and Weitsman (1966) and Oden et al. (1970).

$$\sigma_{111} = (\lambda + 2\mu)\bar{l}^2 u_{1,11} \tag{37}$$

and the following relationship between couple stress and rotation gradient was used by Fleck and Huchinson (1993).

$$\sigma_{i[jk]} = \mu \bar{l}^2 u_{[k,j]i} \tag{38}$$

The value $\bar{l}$ is a material length parameter.

By comparing Eq. (36) with the derived stress–strain relationship in Eq. (34), the present model shows that, for a granular media, the corresponding constants can be determined from Lame constants and internal length as follows:

$$a_1 = \frac{L^2}{6}\lambda; \quad a_2 = a_3 = 0; \quad a_4 = \frac{L^2}{6}(\mu + \alpha); \quad a_5 = \frac{L^2}{6}(\mu - \alpha) \tag{39}$$

These constants can also be determined in terms of inter-particle stiffness (see Eq. (28)),

$$a_1 = a_5 = \frac{Nl^2}{15V}\frac{L^2}{6}(k_n - k_s); \quad a_4 = \frac{Nl^2}{15V}\frac{L^2}{6}(k_n + 4k_s) \tag{40}$$

For the case of $k_s = 0$, all three constants are equal, i.e., $a_1 = a_4 = a_5 = \frac{Nl^2}{15V}\frac{L^2}{6}k_n$

3.3. Two deformation modes: torsion and bending

In the present model, Eq. (34) represents 18 equations, which can be decomposed into two fully decoupled modes: a torsion mode and a bending mode. The torsion mode contains three equations while the bending mode contains 15 equations. In terms of three constants λ, μ and L, the three equations of torsion mode and 15 equations of bending mode are given below in matrix forms.

(1) Torsion mode (3 equations)

$$\begin{Bmatrix} \sigma_{123} \\ \sigma_{132} \\ \sigma_{231} \end{Bmatrix} = \frac{L^2}{6}\begin{bmatrix} 4\mu - 2\lambda & \lambda & \lambda \\ \lambda & 4\mu - 2\lambda & \lambda \\ \lambda & \lambda & 4\mu - 2\lambda \end{bmatrix}\begin{Bmatrix} u_{3,12} \\ u_{2,31} \\ u_{1,23} \end{Bmatrix} \tag{41}$$

(2) Bending mode (15 equations)

The 15 equations can be expressed into three sets of fully decoupled equations; each set contains five equations. The first set of five equations is given by

$$\begin{Bmatrix} \sigma_{111} \\ \sigma_{122} \\ \sigma_{133} \\ \sigma_{221} \\ \sigma_{331} \end{Bmatrix} = \frac{L^2}{12}\begin{bmatrix} 4(2\mu + \lambda) & 2\lambda & 2\lambda & 0 & 0 \\ 2\lambda & 4\mu & \lambda & 2\lambda & 0 \\ 2\lambda & \lambda & 4\mu & 0 & 2\lambda \\ 0 & 2\lambda & 0 & 4(2\mu - \lambda) & 0 \\ 0 & 0 & 2\lambda & 0 & 4(2\mu - \lambda) \end{bmatrix}\begin{Bmatrix} u_{1,11} \\ 2u_{2,21} \\ 2u_{3,31} \\ u_{1,22} \\ u_{1,33} \end{Bmatrix} \tag{42}$$

The second set of five equations can be obtained from Eq. (42) by replacing the indices of σ_{ijk} and $u_{q,lm}$ in a rotating manner ($1 \to 2$, $2 \to 3$, $3 \to 1$). Repeating the process, the third set of five equations can be obtained.

4. Elastic material constants for couple stress continuum

The stress–strain relationship derived in the previous section consists of all components of strain-gradients. It will be seen in this section that, by neglecting some components of the strain gradients, the stress–strain relationship can be simplified to a form that resembles the couple-stress model advocated by Toupin (1962), Mindlin and Tiersten (1962), Fleck and Huchinson (1993), among others. Therefore a couple-stress model can indeed be considered as a special case of the strain-gradient model. For application purposes, the couple-stress model is more popular than the strain-gradient model because of its simplicity. However, it shares the same problem of strain-gradient model that the constitutive constants are difficult to be determined in laboratory. Therefore, in the following, the constitutive constants are derived specifically for a couple-stress model.

In a couple-stress model, the work done is generated only from the gradients of rigid-body rotation. Since the rigid-body rotation is defined as the skew-symmetric part of strain, $\phi_i = \frac{1}{2}e_{ijk}u_{k,j}$, the rotation gradient can be related to displacement gradient by

$$\chi_{ip} = \phi_{i,p} = \tfrac{1}{2}e_{ijk}u_{k,jp} \tag{43}$$

The rotation gradient χ_{ij} contains nine components. The expanded equations for all components of rotation gradients are shown in Appendix B. It is noted that the following relation is always true $\chi_{11} + \chi_{22} + \chi_{33} = 0$. Therefore, the rotation gradients can only be considered as eight independent terms. Among the nine components, three components belong to the torsion mode (χ_{ij}; $i = j$) and six components belong to the bending mode (χ_{ij}; $i \neq j$).

The work done for a couple-stress continuum is expressed as

$$W = \sigma_{(ij)}u_{(j,i)} + m_{ij}\chi_{ji} \tag{44}$$

In this equation, the couple stress m_{ij} is the energy counter-part of rotation gradient χ_{ij}, which can be expressed as higher-order stress as follows (see Appendix C):

$$m_{ij} = e_{jkl}\bar{\sigma}_{ikl} \tag{45}$$

where $\bar{\sigma}_{ijk}$ is defined as: $\bar{\sigma}_{ijk} = \sigma_{ijk} + \sigma_{jik}$ (for $i \neq j$) and $\bar{\sigma}_{ijk} = \sigma_{ijk}$ (for $i = j$).

4.1. Stress–strain relationship

By substituting Eq. (34) into Eq. (45), m_{ij} can be derived as a function of displacement gradients $u_{l,ki}$. Then each term of the displacement gradients $u_{l,ki}$ can be separated into a skew-symmetric part χ_{kl} and a symmetric part $\bar{\chi}_{kl}$ (see Appendix B). After neglecting the symmetric part $\bar{\chi}_{kl}$, we obtain the stress–strain relationship between m_{ij} and χ_{kl}. The final stress–strain relationship can then be written as a tensor form

$$m_{ij} = a_{ijkl}\chi_{lk} \tag{46}$$

where the stiffness tensor

$$a_{ijkl} = \frac{L^2}{3}\left(3(2\mu - \lambda)\delta_{ik}\delta_{jl} - \lambda\delta_{il}\delta_{jk}\right) \tag{47}$$

Eq. (46) is the stress–strain relationship for a couple-stress continuum, which has a form similar to that used in micropolar (or Cosserat) continuum advocated by Cosserat and Cosserat (1909), Gunther (1958), Koiter (1964), Mindlin (1969), Eringen (1968), Shaefer (1967), etc. Nevertheless, a couple-stress model is conceptually different from a micropolar model. In a micropolar model, the stress σ_{ij} and the strain $u_{i,j}$ need not be symmetric while in a couple-stress model the stress and strain are always symmetric. It is further noted that the derived material constants in Eq. (47) are different from the material constants derived for granular material treated as a micropolar continuum by Chang and Ma (1992). In a micropolar continuum, the material constants are related to the rolling stiffness between particles, which is neglected in the derivation of the present model.

The stress–strain relationship of Eq. (46) can be decomposed into two fully-decoupled modes: the torsion mode, and the bending mode. In order to discuss the material constants associated with each mode, the stress–strain relationship for both modes are written in the following matrix form:

(1) Torsion mode

$$\begin{Bmatrix} m_{11} \\ m_{22} \\ m_{33} \end{Bmatrix} = \begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} \begin{Bmatrix} \chi_{11} \\ \chi_{22} \\ \chi_{33} \end{Bmatrix} \tag{48}$$

The value of torsion stiffness T can be expressed in terms of λ and μ as

$$T = \frac{2L^2}{3}(4\mu - 3\lambda) \tag{49}$$

In terms of inter-particle stiffness, the torsion stiffness is

$$T = \frac{Nl^2}{15V}\frac{2}{3}L^2(k_n + 9k_s) \tag{50}$$

(2) Bending mode

$$\begin{Bmatrix} m_{12} \\ m_{21} \\ m_{13} \\ m_{31} \\ m_{23} \\ m_{32} \end{Bmatrix} = \begin{bmatrix} \kappa & -2a_1 & 0 & 0 & 0 & 0 \\ -2a_1 & \kappa & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa & -2a_1 & 0 & 0 \\ 0 & 0 & -2a_1 & \kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa & -2a_1 \\ 0 & 0 & 0 & 0 & -2a_1 & \kappa \end{bmatrix} \begin{Bmatrix} \chi_{21} \\ \chi_{12} \\ \chi_{31} \\ \chi_{13} \\ \chi_{32} \\ \chi_{23} \end{Bmatrix} \tag{51}$$

where $2a_1 = \frac{L^2}{3}\lambda$ and

$$\kappa = \frac{L^2}{2}(2\mu - \lambda) \tag{52}$$

In terms of inter-particle stiffness, the bending stiffness is

$$\kappa = \frac{Nl^2}{15V}\frac{1}{2}L^2(k_n + 4k_s) \tag{53}$$

The physical meanings of couple-stress and rotation gradient are illustrated here. In a Cartesian coordinate system (x_1, x_2, x_3), the couple-stress m_{ij} acts on face-x_i in the direction of x_j. The couple-stress is in torsion when $i = j$, and in bending when $i \neq j$. An example of m_{12} is shown in Fig. 3a (the axis x_2 is in the direction inward to the 1–3 plane) and m_{21} in Fig. 3b (the axis x_1 is in the direction outward from the 2–3 plane).

Deformation of the corresponding rotation gradient χ_{21} is also shown in Fig. 3a. It is noted that, according to its definition in Eq. (B.1) (see Appendix B), a positive χ_{21} indicates a negative curvature $u_{3,11}$ along x_1-direction. On the other hand, a positive rotation gradient χ_{12} shown in Fig. 3b indicates a positive curvature $u_{3,22}$ along x_2-direction.

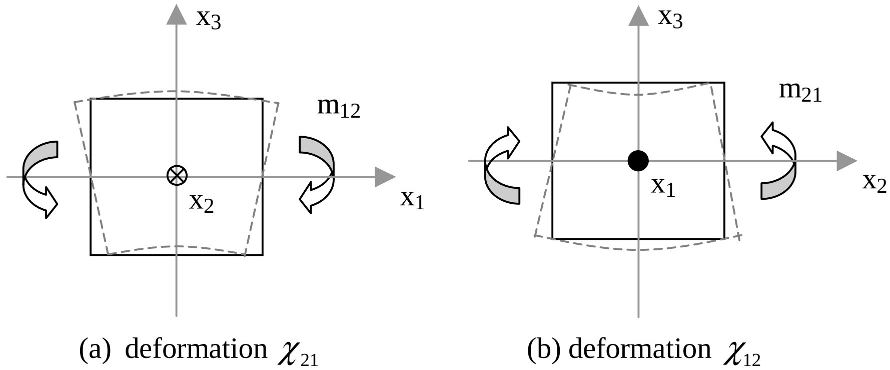

Fig. 3. Illustration of couple stress and rotation gradient: (a) deformation χ_{21} under m_{12}, and (b) deformation χ_{21} under m_{12}.

4.2. Effect of inter-particle stiffness on material constants

For an example of pure bending, the applied couple stress is m_{12}. All the other components of the couple stress m_{ij} are zero. According to Eq. (51), only two rotation gradients (χ_{12} and χ_{21}) are affected by the application of m_{12}. The ratio of the two rotation gradients is

$$\frac{\chi_{12}}{\chi_{21}} = \frac{\lambda}{6\mu - 3\lambda} \tag{54}$$

In terms of inter-particle stiffness, the ratio of rotation gradient becomes

$$\frac{\chi_{12}}{\chi_{21}} = \frac{k_n - k_s}{3k_n + 4k_s} \tag{55}$$

Based on Eq. (28), it is noted that the ratio of rotation gradients χ_{12}/χ_{21} is equal to Poisson's ratio. For the usual material, the normal inter-particle stiffness is usually greater than the shear inter-particle stiffness (i.e., $k_n > k_s$). According to Eq. (55), the ratio χ_{12}/χ_{21} is positive. As previously shown in Fig. 3a and b, the deformation pattern exhibits a negative longitudinal curvature $u_{3,11}$ and a positive transverse curvature $u_{3,22}$. Thus under a bending of the volume element, the longitudinal curvature and the transverse curvature carry opposite signs, representing a shape of saddle with an anticlastic surface as shown in Fig. 4.

Curvature of the anticlastic surface depends on the ratio of rotation gradients χ_{12}/χ_{21}. It can be seen that under the special case of $k_s = k_n$ (i.e., Poisson's ratio $= 0$), the transverse curvature equals to zero ($\chi_{12} = 0$). Under the special case of $k_s = 0$ (i.e., Poisson's ratio $= 1/3$), the transverse curvature is 1/3 of the longitudinal curvature ($\chi_{12} = \chi_{21}/3$). The ratio of rotation gradients versus the ratio of k_s/k_n is plotted in Fig. 5.

The ratio of torsion stiffness or bending stiffness to shear modulus has a dimension of length square. It is convenient to define constants c_T and c_B such that the torsion and bending stiffness associated with the higher-order strain can be linked to the shear modulus by

$$T = c_T \mu L^2; \quad \kappa = c_B \mu L^2 \tag{56}$$

where the constant c_T and c_B are functions of inter-particle stiffness, given by

$$c_T = \frac{4}{3}\left(\frac{k_n + 9k_s}{2k_n + 3k_s}\right) \tag{57}$$

$$c_B = \frac{k_n + 4k_s}{2k_n + 3k_s} \tag{58}$$

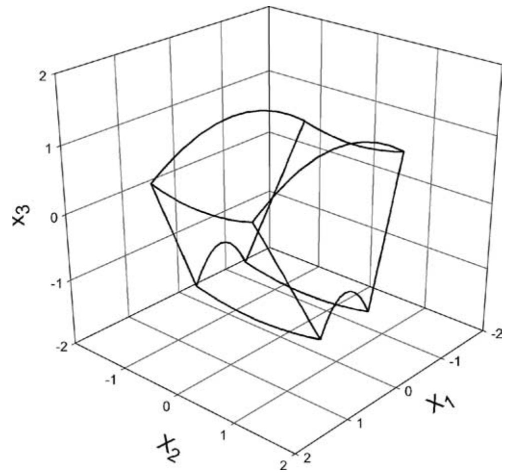

Fig. 4. Anticlastic surface.

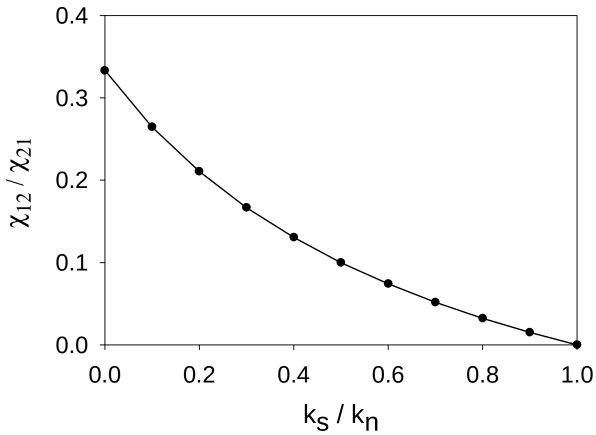

Fig. 5. Ratio of curvature of the anticlastic surface.

Eqs. (57) and (58) are plotted on Figs. 6 and 7. It can be observed that in the range of k_s from 0 to k_n, the constant c_B ranges from 0.5 to 1, and the constant c_T ranges from 0.66 to 2.66. Under a special case of $k_s = k_n$ (i.e., Poisson's ratio = 0), the magnitude of torsion stiffness is 2.66 times of bending stiffness. Under a special case of $k_s = 0$ (i.e., Poisson's ratio = 1/3), the torsion stiffness and the bending stiffness are about the same in magnitude.

5. Summary and conclusion

We have derived the stress–strain relationship for a representative element of granular material by taking into account the inter-particle stiffness and fabric tensor of the packing. By using a polynomial displacement field containing second displacement gradients, the derived model resembles the first-order "strain-gradient continua" within Mindlin–Toupin's framework.

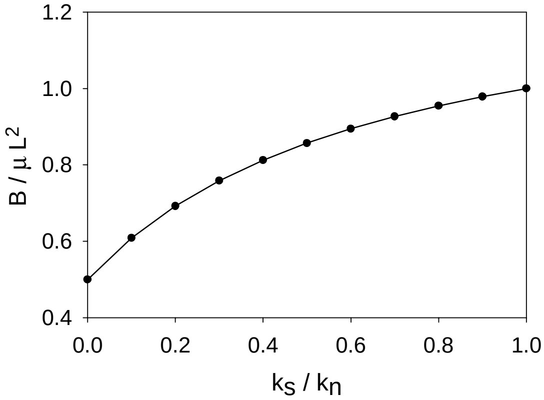

Fig. 6. Bending stiffness as functions of inter-particle stiffness.

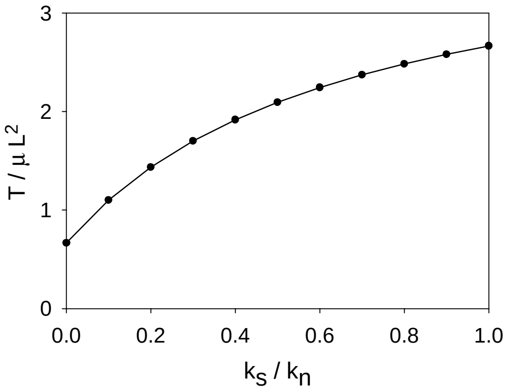

Fig. 7. Torsion stiffness as functions of inter-particle stiffness.

One of the difficulties in using ‘‘strain-gradient model’’ is that the material constants cannot be determined from the commonly used laboratory tests. For this purpose, we have derived closed-form expressions of elastic constants for random packing. It is noted that the total number of material constants for the fourth rank tensor C_{ijkl} and the sixth rank tensor D_{ijklmn} is 810. The derived results show that, for an isotropic random packing of granulates, the constitutive tensor C_{ijkl} and D_{ijklmn} can be determined from five independent constants: inter-particle stiffness k_n, k_s, particle size l, packing density N/V, and the size of the representative volume L.

Another useful result from this derivation is that the material constants associated with higher-order stress–strain relationship D_{ijklmn} are directly related to the lower-order material constants λ, μ, and internal length L. Thus the higher-order constants can be estimated using the measured values of the usual material constants λ and μ. These results can also be used to determine the five parameters proposed by Mindlin and Eshel (1968).

With some restrictions, the ‘‘strain-gradient continuum’’ can be simplified to a subclass of ‘‘couple- stress solid’’ and the 18 higher-order stress–strain equations are reduced to nine. These nine equations can be fully decoupled into two modes: three equations in torsion and six equations in bending. The derived closed-form expressions of elastic constants for the couple-stress solid show that the bending stiffness and torsion stiffness are influenced by the magnitude of inter-particle stiffness k_s and k_n. When the value of k_s increases from 0 to k_n, the bending stiffness increases about two times and torsion stiffness increases about four times.

Appendix A. Separate a summation based on covariance theory

The covariance of two random variables, X and Y, can be defined as follows: (Benjamin and Cornell, 1970)

$$\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y] \tag{A.1}$$

where $E[X]$, $E[Y]$, and $E[XY]$ represents the expectations of X, Y and XY respectively. When X and Y are not correlated, the covariance becomes zero and the expectation of the product XY is equal to the product of the expectation of X and the expectation of Y.

Now, we let X be the set of variables of $x_j^c x_m^c$ and let Y be the set of variables of $l_i^c l_l^c K_{pk}^c$. By definition, the expectations of X, Y, and XY are given by

$$E[X] = \frac{1}{N}\sum_{c=1}^{N} x_j^c x_m^c \tag{A.2}$$

$$E[Y] = \frac{1}{N}\sum_{c=1}^{N} l_i^c l_l^c K_{pk}^c \tag{A.3}$$

$$E[XY] = \frac{1}{N}\sum_{c=1}^{N} x_j^c x_m^c l_i^c l_l^c K_{pk}^c \tag{A.4}$$

Since the locations ($x_j^c x_m^c$) of inter-particle contact are assumed not to correlate with the branch length and contact properties ($l_i^c l_l^c K_{pk}^c$), the expectation of the product of XY is equal to the product of expectation according to Eq. (A.1), thus

$$\frac{1}{N}\sum_{c=1}^{N} x_j^c x_m^c l_i^c l_l^c K_{pk}^c = \left(\frac{1}{N}\sum_{c=1}^{N} x_j^c x_m^c\right)\left(\frac{1}{N}\sum_{c=1}^{N} l_i^c l_l^c K_{pk}^c\right) \tag{A.5}$$

Appendix B. Definition of gradients for rigid-body rotation and shear strain

For the purpose of modeling couple stress continua, we decompose the strain gradients into symmetric and skew-symmetric parts. According to the definition given in Eq. (43), the nine components for the gradient of skew-symmetric strain (i.e., rotation gradient) χ_{ip} are expressed below.

$$\begin{aligned}
&\chi_{11} = \tfrac{1}{2}(u_{3,21} - u_{2,31}); \quad \chi_{22} = \tfrac{1}{2}(u_{1,32} - u_{3,12}); \quad \chi_{33} = \tfrac{1}{2}(u_{2,13} - u_{1,23}) \\
&\chi_{12} = \tfrac{1}{2}(u_{3,22} - u_{2,32}); \quad \chi_{13} = \tfrac{1}{2}(u_{3,23} - u_{2,33}); \quad \chi_{23} = \tfrac{1}{2}(u_{1,33} - u_{3,13}) \\
&\chi_{21} = \tfrac{1}{2}(u_{1,31} - u_{3,11}); \quad \chi_{31} = \tfrac{1}{2}(u_{2,11} - u_{1,21}); \quad \chi_{32} = \tfrac{1}{2}(u_{2,12} - u_{1,22})
\end{aligned} \tag{B.1}$$

Corresponding to the rotation gradients, nine corresponding terms for the gradients of symmetric shear strain are given below.

$$\begin{aligned}
&\bar{\chi}_{11} = \tfrac{1}{2}(u_{3,21} + u_{2,31}); \quad \bar{\chi}_{22} = \tfrac{1}{2}(u_{1,32} + u_{3,12}); \quad \bar{\chi}_{33} = \tfrac{1}{2}(u_{2,13} + u_{1,23}) \\
&\bar{\chi}_{12} = \tfrac{1}{2}(u_{3,22} + u_{2,32}); \quad \bar{\chi}_{13} = \tfrac{1}{2}(u_{3,23} + u_{2,33}); \quad \bar{\chi}_{23} = \tfrac{1}{2}(u_{1,33} + u_{3,13}) \\
&\bar{\chi}_{21} = \tfrac{1}{2}(u_{1,31} + u_{3,11}); \quad \bar{\chi}_{31} = \tfrac{1}{2}(u_{2,11} + u_{1,21}); \quad \bar{\chi}_{32} = \tfrac{1}{2}(u_{2,12} + u_{1,22})
\end{aligned} \tag{B.2}$$

Among the 18 terms, 12 of them belong to bending mode (i.e., χ_{ij} and $\bar{\chi}_{ij}$ when $i \neq j$) and six of them belong to torsion mode (i.e., χ_{ij} and $\bar{\chi}_{ij}$ when $i = j$).

For the bending mode, the 12 variables χ_{12}, χ_{21}, χ_{13}, χ_{31}, χ_{23}, χ_{32}, and $\bar{\chi}_{12}$, $\bar{\chi}_{21}$, $\bar{\chi}_{13}$, $\bar{\chi}_{31}$, $\bar{\chi}_{23}$, $\bar{\chi}_{32}$ have one-to-one correspondence to the 12 terms of displacement gradients ($u_{3,22}$, $u_{2,32}$, $u_{3,23}$, $u_{2,33}$, $u_{1,33}$, $u_{3,13}$, $u_{1,31}$, $u_{3,11}$, $u_{2,11}$, $u_{1,21}$, $u_{2,12}$, $u_{1,22}$), given by

$$\begin{aligned} u_{2,11} &= \bar{\chi}_{31} + \chi_{31} \quad & u_{3,11} &= \bar{\chi}_{21} - \chi_{21} \quad & u_{3,22} &= \bar{\chi}_{12} + \chi_{12} \\ u_{1,21} &= \bar{\chi}_{31} - \chi_{31} & u_{1,31} &= \bar{\chi}_{21} + \chi_{21} & u_{2,32} &= \bar{\chi}_{12} - \chi_{12} \\ u_{2,12} &= \bar{\chi}_{32} + \chi_{32} & u_{3,13} &= \bar{\chi}_{23} - \chi_{23} & u_{3,23} &= \bar{\chi}_{13} + \chi_{13} \\ u_{1,22} &= \bar{\chi}_{32} - \chi_{32} & u_{1,33} &= \bar{\chi}_{23} + \chi_{23} & u_{2,33} &= \bar{\chi}_{13} - \chi_{13} \end{aligned} \tag{B.3}$$

For the torsion mode, the six terms χ_{11}, χ_{22}, χ_{33}, $\bar{\chi}_{11}$, $\bar{\chi}_{22}$, $\bar{\chi}_{33}$ are related to only three terms of displacement gradients $u_{2,13}$, $u_{1,23}$, $u_{3,12}$. Therefore, $u_{i,jk}$ is not uniquely related to the terms of χ_{ij} and $\bar{\chi}_{ij}$. For example, the displacement gradient $u_{3,12} = \chi_{11} + \bar{\chi}_{11}$ and it can also be expressed as $u_{3,12} = \bar{\chi}_{22} - \chi_{22}$. For convenience, we adopt the following relationships:

$$\begin{aligned} u_{3,12} &= \chi_{11} - \chi_{22} + \bar{\chi}_{33} \\ u_{1,23} &= \chi_{22} - \chi_{33} + \bar{\chi}_{11} \\ u_{2,31} &= \chi_{33} - \chi_{11} + \bar{\chi}_{22} \end{aligned} \tag{B.4}$$

Appendix C. The relationship between couple stress and higher-order stress

The expression of work done by strain gradient is

$$W = \sigma_{ijk} u_{k,ij} \tag{C.1}$$

This represents a summation of 27 terms. Since the subscripts i and j are reciprocal (i.e., $u_{k,ij} = u_{k,ji}$), the 27 terms can be reduced to 18 independent terms, given by

$$W = \bar{\sigma}_{ijk} u_{k,ij} \tag{C.2}$$

where $\bar{\sigma}_{ijk}$ represents 18 terms and is defined as $\bar{\sigma}_{ijk} = \sigma_{ijk} + \sigma_{jik}$ (for $i \neq j$) and $\bar{\sigma}_{ijk} = \sigma_{ijk}$ (for $i = j$). The 18 terms of work done can be separate into three modes, namely,

(1) dilation mode (3 terms),

$$\bar{\sigma}_{111} u_{1,11} + \bar{\sigma}_{222} u_{2,22} + \bar{\sigma}_{333} u_{3,33} \tag{C.3}$$

(2) torsion mode (3 terms),

$$\bar{\sigma}_{123} u_{3,21} + \bar{\sigma}_{132} u_{2,31} + \bar{\sigma}_{231} u_{1,32} \tag{C.4}$$

(3) bending mode (12 terms),

$$\begin{aligned} &\bar{\sigma}_{112} u_{2,11} + \bar{\sigma}_{113} u_{3,11} + \bar{\sigma}_{121} u_{1,21} + \bar{\sigma}_{122} u_{2,21} + \bar{\sigma}_{131} u_{1,31} + \bar{\sigma}_{133} u_{3,31} + \bar{\sigma}_{221} u_{1,22} + \bar{\sigma}_{223} u_{3,22} + \bar{\sigma}_{232} u_{2,32} \\ &\quad + \bar{\sigma}_{233} u_{3,32} + \bar{\sigma}_{331} u_{1,33} + \bar{\sigma}_{332} u_{2,33} \end{aligned} \tag{C.5}$$

By substituting Eqs. (B.3) and (B.4) into Eq. (C.2) and neglecting the work done by dilation mode, the work done can be expressed in terms of χ_{ip} and $\bar{\chi}_{ip}$, given by

$$W = m_{pi} \chi_{ip} + \bar{m}_{pi} \bar{\chi}_{ip} \tag{C.6}$$

In a couple-stress model (see Toupin, 1962; Mindlin and Tiersten, 1962; Fleck and Huchinson, 1993, among others), the work done by $\bar{\chi}_{ip}$ is further neglected, thus

$$W = m_{pi}\chi_{ip}$$
$$= (\bar{\sigma}_{223} - \bar{\sigma}_{232})\chi_{12} - (\bar{\sigma}_{113} - \bar{\sigma}_{131})\chi_{21} - (\bar{\sigma}_{332} - \bar{\sigma}_{233})\chi_{13} + (\bar{\sigma}_{112} - \bar{\sigma}_{121})\chi_{31} + (\bar{\sigma}_{331} - \bar{\sigma}_{133})\chi_{23}$$
$$- (\bar{\sigma}_{221} - \bar{\sigma}_{122})\chi_{32} + (\bar{\sigma}_{123} - \bar{\sigma}_{132})\chi_{11} + (\bar{\sigma}_{231} - \bar{\sigma}_{123})\chi_{22} + (\bar{\sigma}_{132} - \bar{\sigma}_{231})\chi_{33} \quad (C.7)$$

The corresponding couple stress m_{ij} can be expressed in an abbreviated notation, given by

$$m_{ij} = e_{jkl}\bar{\sigma}_{ikl} \quad (C.8)$$

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